DOUBLE SHAPE INVARIANCE OF TWO-DIMENSIONAL SINGULAR MORSE MODEL

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A second shape invariance property of the two-dimensional generalized Morse potential is discovered. Though the potential is not amenable to conventional separation of variables, the above property allows to build purely algebraically part of the spectrum and corresponding wave functions, starting from *one* definite state, which can be obtained by the method of *SUSY*-separation of variables, proposed recently.

1.   Introduction

Recently the notion of shape invariance [1] was generalized [2], [3] to two-dimensional cases, which are not amenable to conventional separation of variables in the Schrödinger equation. It was shown that, in contrast to the one-dimensional situation, in general the shape invariance itself gives algebraically *only part* of spectra (and wave functions), leading to the partial (quasi-exact) solvability of the model. The fact that only a partial solution for the spectral problem is provided by shape invariance for two-dimensional problems is related to the dependence (in general) of the **ground state energy** on the parameters of the model (in one-dimensional case one usually had $E_{gs} \equiv 0$, not depending on the parameter of shape invariance). Also, possible degeneracy of levels in two-dimensional models is important: for complete solvability one has to know both energy levels and **all** corresponding wave functions.

In particular, the two-dimensional generalized (singular[d]) Morse potential was demon-

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[d]We remark that the behaviour of wave functions at the singularity is under control (see [2]).

strated [2], [3] to satisfy shape invariance. Furthermore this model allows for *SUSY-*separation of variables (i.e. separation of variables in the supercharge), and therefore makes it possible to construct some wave functions as linear combinations of zero modes of the supercharge. Then each of these "principal" eigenfunctions was used to generate (purely algebraically!) the shape invariance chain of excited levels.

In the present paper some additional properties of the two-dimensional generalized Morse potential are studied (Section 2). It is shown that besides the shape invariance, described in [2], the model possesses an additional shape invariance, which can also be used to construct corresponding chains of states (Section 3). In this construction the wave functions of the **first** shape invariance chain of the previous Section play the role of the "principal" states for new shape invariance chains. Thus it is shown (Section 4) that the combination of both shape invariances allows to build **algebraically** the same part of the spectrum and corresponding eigenfunctions starting from **only one** "principal" state. The mutual interrelation between the "old" and "new" chains is also clarified.

2. Shape invariant two-dimensional Morse model

The direct d-dimensional ($d \geq 2$) generalization of Witten's SUSY QM includes [4] Schrödinger operators both with scalar and with matrix potentials. In order to avoid the appearance of matrix components in the two-dimensional Superhamiltonian, one can explore [5], [3] for the case $d = 2$ the idea of one-dimensional Higher Order SUSY QM [6] and to take, for example, the second order supercharges with hyperbolic (Lorentz) metrics:

$$Q^{\pm} = \partial_1^2 - \partial_2^2 + C_i(\vec{x})\partial_i + B(\vec{x}); \qquad \vec{x} = (x_1, x_2); \qquad \partial_i = \partial/\partial x_i; \quad i = 1, 2. \qquad (1)$$

The main relations of SUSY QM are the two-dimensional SUSY intertwining relations:

$$H^{(1)}Q^{+} = Q^{+}H^{(2)}; \qquad H^{(1,2)} = -\Delta^{(2)} + V^{(1,2)}(\vec{x}); \quad \Delta^{(2)} \equiv \partial_i^2, \qquad (2)$$

for which some partial solutions were found [5] providing nontrivial models with Schrödinger Hamiltonians $H^{(1)}, H^{(2)}$, not allowing separation of variables. It is convenient to consider also light-cone coordinates:

$$x_{\pm} \equiv x_1 \pm x_2; \qquad \partial_{\pm} = \frac{1}{2}(\partial_1 \pm \partial_2); \quad C_{\pm} = C_1 \mp C_2, \qquad (3)$$

where [5] due to (2) $C_\pm = C_\pm(x_\pm)$, and

$$\partial_-(C_-F) = -\partial_+(C_+F); \qquad F(\vec{x}) \equiv F_1(2x_1) + F_2(2x_2). \tag{4}$$

The potentials $V^{(1,2)}(\vec{x})$ are:

$$V^{(1,2)}(\vec{x}) = \pm\frac{1}{2}\Big(C'_+(x_+) + C'_-(x_-)\Big) + \frac{1}{8}\Big(C^2_+(x_+) + C^2_-(x_-)\Big) + \frac{1}{4}\Big(F_2(2x_2) - F_1(2x_1)\Big), \tag{5}$$

and

$$B(\vec{x}) = \frac{1}{4}\Big(C_+C_- + F_1(2x_1) + F_2(2x_2)\Big). \tag{6}$$

Recently two specific models of the list of particular solutions of (4) were shown [2], [7] to be partially (quasi-exactly [8]) solvable, i.e. part of eigenvalues and eigenfunctions of the system was found analytically.

The first of the models - the two-dimensional Morse potential - is characterized[e] by:

$$C_+(x_+) \;=\; 4a\alpha; \qquad C_-(x_-) = 4a\alpha\coth(\frac{\alpha x_-}{2}); \tag{7}$$

$$\frac{1}{4}F_{1,2}(2x_{1,2}) \;=\; \mp(2c - \alpha)\exp(-\alpha x_{1,2}) \mp \exp(-2\alpha x_{1,2}); \tag{8}$$

With this choice the potentials (5) can be naturally interpreted as a two-dimensional (non-separable, singular) generalization of the one-dimensional Morse potential:

$$\begin{aligned} V^{(1,2)}(\vec{x}; a, \alpha, c) \;=\;& \alpha^2 a(2a \mp 1)\sinh^{-2}(\alpha x_-/2) + (2c - \alpha)\Big(\exp(-\alpha x_1) + \exp(-\alpha x_2)\Big) + \\ & + \Big(\exp(-2\alpha x_1) + \exp(-2\alpha x_2)\Big) + 4a^2\alpha^2. \end{aligned} \tag{9}$$

While the Schrödinger equations with potential $V^{(1,2)}$ in (9) are not amenable to standard separation of variables, nevertheless, we can apply the recently proposed [2] (see also [3]) method of $SUSY$-separation of variables (variables are separable not in $H^{(1,2)}$, but in the supercharge Q^+). Then zero modes of Q^+ were constructed analytically in terms of hypergeometric functions: suitably chosen linear combinations of them provide (see details in [2], [3]) a set of "principal" wave functions $\Psi_{k,0}^{(2)}(\vec{x}; a, \alpha, c)$ of the Hamiltonian $H^{(2)}(\vec{x}; a, \alpha, c)$ and corresponding energy eigenvalues, which depend on arbitrary values of parameters:

$$E_{k,0}^{(2)}(a, \alpha, c) = -2\Big[2a\alpha^2 s_k - \epsilon_k\Big] = -2[-2a\alpha^2(\frac{c}{\alpha} + k) + (c + k\alpha)^2]; \quad k = 0, 1, 2... \tag{10}$$

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[e]The constants used here differ from those in [2], [3]. The present choice is useful for the derivation below and can be easily made consistent with the previous ones by suitable shifts of coordinates.

ϵ_k and s_k above were defined in [2], and are now expressed in terms of new parameter c instead of $A \equiv (c - \frac{\alpha}{2})^2$ (the appropriate constant shift of $x_{1,2}$ was also used here).

The model of Eq.(9) enjoys [2] an additional remarkable property - the two-dimensional shape invariance:

$$H^{(1)}(\vec{x}; a, \alpha, c) = H^{(2)}(\vec{x}; a - \frac{1}{2}, \alpha, c) + \mathcal{R}(a, \alpha, c); \quad \mathcal{R}(a, \alpha, c) = \alpha^2(4a - 1). \quad (11)$$

Similarly to the one-dimensional shape invariance, each "principal" eigenfunction $\Psi_{k,0}^{(2)}$ gives start to a whole shape invariance chain of eigenstates of $H^{(2)}(\vec{x}; a, \alpha, c)$, which can be built by means of a sequence of supercharges $Q^- \equiv (Q^+)^\dagger$:

$$\Psi_{k,m}^{(2)}(\vec{x}; a, \alpha, c) = Q^-(a, \alpha, c) \cdot Q^-(a - \frac{1}{2}, \alpha, c) \cdot Q^-(a - 1, \alpha, c) \cdot ...$$

$$... \cdot Q^-(a - \frac{1}{2}(m - 1), \alpha, c) \Psi_{k,0}^{(2)}(\vec{x}; a - \frac{1}{2}m, \alpha, c); \quad (m = 1, 2, ...), \quad (12)$$

with eigenvalues:

$$E_{k,m}^{(2)}(a, \alpha, c) = E_{k,0}^{(2)}(a - \frac{1}{2}m, \alpha, c) + \mathcal{R}(a - \frac{1}{2}(m - 1), \alpha, c) + ... + \mathcal{R}(a, \alpha, c) =$$

$$= 2(c + k\alpha)\Big(2a\alpha - a\alpha m - (c + k\alpha)\Big) + \alpha^2 m(4a - m). \quad (13)$$

The sequence in (12) is constrained only by the condition of normalizability of these wave functions.

3.   The new shape invariance of the Morse potential

In order to demonstrate the existence of a second invariance of the two-dimensional Morse potential (9), we link it to another two-dimensional system, found in [5] among solutions (4) - (6) of intertwining relations with hyperbolic (Lorentz) metrics in supercharges. The corresponding functions will be denoted by tildes and their arguments by $\vec{y} = (y_1, y_2)$:

$$\tilde{C}_+(y_+) = 4\Big(\exp(\alpha y_+) + c\Big); \quad \tilde{C}_-(y_-) = 4\Big(\exp(\alpha y_-) + c\Big); \quad y_\pm \equiv y_1 \pm y_2$$

$$\tilde{F}_1(2y_1) = 0; \quad \frac{1}{4}\tilde{F}_2(2y_2) = 2d\Big(\exp(\alpha y_2) - \exp(-\alpha y_2)\Big)^{-2};$$

$$\tilde{B} = 4\Big(\exp(\alpha y_-) + c\Big) \cdot \Big(\exp(\alpha y_+) + c\Big) + 2d \sinh^{-2}(\alpha y_2).$$

The superpartner potentials found in [5]:

$$\tilde{V}^{(1,2)}(\vec{y}; a, \alpha, c) = 2\Big(\exp(2\alpha y_+) + \exp(2\alpha y_-)\Big) + 2(2c \pm \alpha)\Big(\exp(\alpha y_+) + \exp(\alpha y_-)\Big)$$

$$+ 2d \cdot \sinh^{-2}(\alpha y_2), \quad (14)$$

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satisfy to the intertwining relations:

$$\widetilde{H}^{(1)}(\vec{y}; a, \alpha, c)\widetilde{Q}^+(\vec{y}; a, \alpha, c) = \widetilde{Q}^+(\vec{y}; a, \alpha, c)\widetilde{H}^{(2)}(\vec{y}; a, \alpha, c);$$

$$\widetilde{Q}^+(\vec{y}; a, \alpha, c) = \partial_{y_1}^2 - \partial_{y_2}^2 + \widetilde{C}_+(y_+)\partial_{y_-} + \widetilde{C}_-(y_-)\partial_{y_+} + \widetilde{B}.$$

In order to relate this system to the model of Section 2, one has to replace:

$$y_+ \equiv -x_1; \quad y_- \equiv -x_2; \qquad d \equiv \alpha^2 a(2a+1);$$

$$\widetilde{H}^{(1,2)}(\vec{y}(\vec{x}); a, \alpha, c) \equiv 2h^{(1,2)}(\vec{x}; a, \alpha, c); \quad \widetilde{Q}^+(\vec{y}(\vec{x}); a, \alpha, c) \equiv q^+(\vec{x}; a, \alpha, c)$$

The components $h^{(1,2)}(\vec{x}; a, \alpha, c)$ of new Superhamiltonian become:

$$h^{(1,2)}(\vec{x}; a, \alpha, c) = -\Delta^{(2)} + \alpha^2 a(2a+1)\sinh^{-2}(\alpha x_-/2) + 4a^2\alpha^2 +$$

$$+ (2c \pm \alpha)\Big(\exp(-\alpha x_1) + \exp(-\alpha x_2)\Big) + \Big(\exp(-2\alpha x_1) + \exp(-2\alpha x_2)\Big) \quad (15)$$

and they are intertwined:

$$h^{(1)}(\vec{x}; a, \alpha, c)q^+(\vec{x}; a, \alpha, c) = q^+(\vec{x}; a, \alpha, c)h^{(2)}(\vec{x}; a, \alpha, c) \tag{16}$$

by the supercharge:

$$q^+(\vec{x}; a, \alpha, c) = 4\partial_1\partial_2 - 4\Big(\exp(-\alpha x_1) + c\Big)\partial_2 - 4\Big(\exp(-\alpha x_2) + c\Big)\partial_1 +$$

$$+ 4\Big(\exp(-\alpha x_1) + c\Big) \cdot \Big(\exp(-\alpha x_2) + c\Big) + 2\alpha^2 a(2a+1)\sinh^{-2}(\alpha x_-/2) \quad (17)$$

From (15) one can conclude that this supersymmetrical system has also the shape invariance property, but in this case in the parameter c:

$$v^{(1)}(\vec{x}; a, \alpha, c) = v^{(2)}(\vec{x}; a, \alpha, c + \alpha) \tag{18}$$

where the term, analogous to $\mathcal{R}$ in (11), now vanishes. Therefore also the spectrum of $h^{(2)}(\vec{x}; a, \alpha, c)$ can be obtained algebraically: starting from some "principal" (for this model) wave function with energy $e_{l,0}^{(2)}(a, \alpha, c)$ $l = 0, 1, 2, ..,$ one will obtain the shape invariance chain of states with energies

$$e_{l,n}^{(2)}(a, \alpha, c) = e_{l,0}^{(2)}(a, \alpha, c + n\alpha) \qquad n = 1, 2, ... \tag{19}$$

As for the choice of "principal" states for this model, it is convenient to choose states of the **first** shape invariance chain of Section 2: $e_{k,0}^{(2)}(a, \alpha, c) = E_{0,k}^{(2)}(a, \alpha, c)$.

Comparing the two supersystems (9) and (15), one notices that:

$$H^{(2)}(\vec{x}; a, \alpha, c) = h^{(2)}(\vec{x}; a, \alpha, c), \tag{20}$$

therefore the same two-dimensional system (9) - with generalized Morse potential - participates in two different intertwining relations (2) and (16) and possesses also two independent shape invariance properties (11) and (18), expressed in transformations of parameters a and c, respectively.

4.   The spectrum of the singular Morse potential

A priori, one has to expect, that each shape invariance will give rise to its own chain of states and corresponding energies. But a more careful analysis shows that these chains include states which are overlapping. First of all, one can observe this overlap from the explicit formulas (13) and (19) for the spectra. But due to the possible (in principle) degeneracy of levels in two-dimensional Quantum Mechanics, it still does not imply the coincidence of wave functions. Nevertheless, this coincidence holds as can be straightforwardly checked by means of the operator equality:

$$q^-(a, \alpha, c) \cdot Q^-(a, \alpha, c + \alpha) = Q^-(a, \alpha, c) \cdot q^-(a - \frac{1}{2}, \alpha, c); \quad q^-(\vec{x}; a, \alpha, c) \equiv (q^+(\vec{x}; a, \alpha, c))^\dagger. \tag{21}$$

For example, just the operators in both sides of this equation, acting onto the wave function $\Psi_{0,0}^{(2)}(a - \frac{1}{2}, \alpha, c + \alpha)$, give *by two different ways* the eigenfunction $\Psi_{1,1}^{(2)}(a, \alpha, c)$ with energy $E_{1,1}^{(2)}(a, \alpha, c)$. Analogously, one can check that the wave function $\Psi_{k,m}^{(2)}$ with arbitrary pair of indices (k, m) can be obtained by different ways, via chains of operators Q^- and q^-, giving the same result due to equalities similar to (21).

A better understanding of the above mentioned overlap can be obtained by realizing that the "principal" states of the new shape invariance (i.e. the states from which one starts the construction of chains) are chosen to be the states $E_{0,m}^{(2)}(a, \alpha, c) = e_{m,0}^{(2)}(a, \alpha, c)$ (see (13)). So, the second shape invariance acts transversely in respect to the first. Acting with $q^-(\vec{x}; a, \alpha, c)$ (see (17)) k times on a generic state $\Psi_{0,m}^{(2)}(a, \alpha, c)$ leads to the state $\Psi_{k,m}^{(2)}$. Thus the overlap can be concisely depicted as

$$E_{k,m}^{(2)} = e_{m,k}^{(2)}. \tag{22}$$

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In other words, one can move first "up" $c \rightarrow c + \alpha$ and then "transversely" $a \rightarrow a - \frac{1}{2}$, or first transversely and then up on the "energy lattice".

5. Remarks and conclusions

We stress that each Hamiltonian, participating in SUSY intertwining relations of second order (2), is integrable [5], [3], i.e. it has a symmetry operator (integral of motion) of fourth order in derivatives:

$$R^{(1)} = Q^+ Q^-; \qquad R^{(2)} = Q^- Q^+; \qquad [H^{(i)}, R^{(i)}] = 0; \quad i = 1, 2, \tag{23}$$

which **cannot** be reduced by elimination of operator functions of the Hamiltonian. It can be checked straightforwardly, though rather tediously, that the symmetry operators associated to intertwining relations (16) *do not* give a new, *independent*, symmetry operator for the Hamiltonian $H^{(2)}$, indeed:

$$q^- q^+ + Q^- Q^+ = 4\left(H^{(2)} + 2c^2\right)^2 + 16c^2(c^2 - 4a^2\alpha^2). \tag{24}$$

The gluing condition (20) of two $SUSY$ systems (9) and (15) provides an opportunity to link the components $H^{(1)}$ and $h^{(1)}$ by intertwining operators of fourth order in derivatives:

$$H^{(1)}\left(Q^+ \cdot q^-\right) = \left(Q^+ \cdot q^-\right) h^{(1)}. \tag{25}$$

This intertwining produces also a shape invariance involving two parameters :

$$H^{(1)}(\vec{x}; a, \alpha, c) = h^{(1)}\left(\vec{x}; a - \frac{1}{2}, \alpha, c - \alpha\right) + \mathcal{R}(a, \alpha, c). \tag{26}$$

The associated symmetry operators for $H^{(1)}$ of order eight $\left(Q^+ \cdot q^-\right) \cdot \left(q^+ \cdot Q^-\right)$ *are reducible* to a polynomial function of $H^{(1)}$ and $R^{(1)}$.

Let us note that treatments, similar to the ones of Sections 2 - 4, can be applied also to the complexified version of singular two-dimensional Morse potential (9) (see [9]), including a complexified version of the related two-dimensional model (14) and leading to a complex form of double shape invariance.

In conclusion, the main results of the paper are the following.

- A new property of two-dimensional Morse model - the second shape invariance - was found and investigated.

- The excited states of new shape invariance chains were proved to coincide with the states obtained from the first shape invariance.

- It was shown that the same set of states of partially solvable two-dimensional Morse model can be built now from **only one** "principal" state $\Psi_{0,0}^{(2)}(a, \alpha, c)$, i.e. one needs **only the first** zero mode of the supercharge Q^+ to be obtained by the method of SUSY-separation of variables. This is much easier to build.

- Though the Morse Hamiltonian (11) obeys two different intertwining relations and has, correspondingly, two fourth order symmetry operators, the system **is not superintegrable**, since these operators are inter-related by the hamiltonian (see (24)).

Acknowledgements

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